Derivable: a Novel Derivatives Pricing Markets based on a Family of Asymptotic Power Curves

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Autumn 2023

Abstract

This paper introduces a novel methodology to construct a decentralized derivatives AMM protocol named Derivable, based on a family of asymptotic power functions. The protocol enables the creation of leveraged derivative markets for any underlying asset, provided price feed. Derivable is completely automated, decentralized and permissionless to all participants: market makers, liquidity providers and long-short traders. Derivable is derived from first principles, inspired by the constant product formula of Uniswap, and utilizes asymptotic power curves to formulate a novel derivatives pricing method for power-perpetual futures. We provide a rigorous mathematical formalization of Derivable and conduct a comprehensive analysis of the model. Our analysis demonstrates that, when properly initialized, the perpetual derivatives tokens are optimally exposed to both sides of power leverages, and the market provides infinite liquidity in all market conditions, rendering it everlasting. Furthermore, the Derivable model represents the first-ever power perpetual market to offer a timely continuous decay mechanism and no liquidation, breaking away from conventional unique position models and order-book in legacy derivatives exchanges.

Key words: AMM, decentralized derivatives, decentralized finance, perpetual futures, power perpetuals.

1 Introduction

Uniswap, https://uniswap.org/, since launch in November 2018, has disrupted traditional spot trading models based on order-book and has laid a solid foundation for decentralized finance (DeFi) that fully leverages on-chain execution. The protocol introduced a novel pricing formula based on inverse functions and developed an innovative spot trading mechanism and automated market-making system that was previously unseen in the financial world. Uniswap's unique value proposition is its infinite liquidity, which is unattainable by traditional trading models. The inception of Uniswap can be traced back to two of Vitalik Buterin's (the creator of Ethereum) posts [4], [5] on https://ethresear.ch and https://www.reddit.com in 2018, where he discussed alternative ways to run decentralized exchanges similar to prediction markets. The first (or fundamental) principles to form any market are

- having buyers and sellers to participate on trades,
- having a pricing mechanism to help buyers and sellers complete trades (this is done by conventional order-book paradigm associated with matching engines utilized on legacy exchanges).

Uniswap followed the fundamental principles but utilizing constant product for pricing mechanism, hence removing order-book model and matching engines.

Recently, derivatives is emerging as a vibrant area in DeFi advancement. Various products for options, futures and perpetuals are introduced (see [6] for a comprehensive landscape by 0xperp). In particular, power perpetuals has been introduced by many DeFi researchers, e.g. Wayne Nilsen [7], and used to hedge impermanent loss [8] for liquidity providers on Uniswap and the like-AMMs. We have studied and been aware that it is challenging to apply legacy derivatives market models on constructing an on-chain efficiently and automatically fully-decentralized derivatives market (see Section 6.2 for existing challenges in derivatives DEXes), in particular, congestion issue

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upon liquidation events due to blockchain's low throughput and gas-intensive limitations. Inspired by Uniswap's groundbreaking invention, we introduce the Derivable protocol, which follows a first-principle approach to disrupt conventional models of derivatives trading and market making based on unique position and order-book exchanges. This paper is considered as a whitepaper of Derivable protocol, which introduces a novel derivatives pricing method based on intuitive asymptotic power curves to construct an automated market-making (AMM) model for decentralized perpetual swap, more explicitly, power-leveraged derivatives trading.

In Section 2, we prove financial meaning of power perpetuals, then formulate pay-off functions for power perpetuals based on a family of asymptotic power functions and analyze their mathematical properties. Then we formalize pricing method for power perpetuals under Black-Scholes assumptions. Based on this mathematically-proven foundation, Section 3 introduce a novel derivatives pricing method and the construction of an innovative decentralized power-perpetuals market paradigm, both derived from asymptotic power curves. Section 4 presents the analysis of Derivable's derivatives market model. The following sections review the evolution of decentralized exchanges, then highlight the most significant characteristics of our model and differentiate Derivable and its unique advantages from existing solutions. The most important contribution of this paper is a novel derivatives pricing method to construct a feasible decentralized power-perpetuals market paradigm.

2 Mathematical Formalization

2.1 Power index

Given a financial derivative, i.e. an index tracking price of a targeted underlying asset (e.g. ETH), without loss of generality. A power index of ETH is denoted as $\mathrm{ETH^k}$, where $k \in \mathbb{Z}$ a non-zero integer. Given a price x of ETH, when it changes $\pm\theta\%$, the power index $\mathrm{ETH^k}$ values $(x \pm \frac{x*\theta}{100})^k$, implying a changing rate of $\left((1 \pm \frac{\theta}{100})^k - 1\right) * 100\%$ for the $\mathrm{ETH^k}$ index. In the following examples, we will see that the power $k > 0$ implies a long index, while $k < 0$ implies a short index.

Examples: If ETH price increases (+1%), then ETH^4 will increases $(1+0.01)^4-1 = +4.06\%$, while ETH^{-4} will decrease $(1+0.01)^{-4}-1 = -3.9\%$. If ETH price decreases (-1%), then ETH^4 will decrease $(1-0.01)^4-1 = -3.94\%$, while ETH^{-4} will increase $(1-0.01)^{-4}-1 = +4.1\%$. Thus, ETH^4 is equivalent to the long-index while ETH^{-4} is for the short-index with approximately 4 times of leverage. Greater power implies higher leverage exposure.

2.2 Financial meaning of power perpetuals

Assume that Alice leverages her trades with a multiplier $k > 1$. We will show that if Alice closes her positions periodically and rolls her wealth into a renewed leveraged position, then her portfolio will behave equivalently as if she buy a power perpetual contract with power k.

Indeed, if her current wealth is Y_t and she borrows $(k-1)Y_t$ to invest into an asset at current price X_t of the underlying asset, then by the end of the trading period, her gain will be kY_tR_t, where $R_t = \frac{dX_t}{X_t}$. We arrives at the equation

$$dY_t = kY_t\frac{dX_t}{X_t} \Rightarrow \frac{dY_t}{Y_t} = k\frac{dX_t}{X_t}.$$

This implies that $Y_t = Y_0X_t^k$ if X_t is of finite variation. If X_t is a Geometric Brownian Motion with volatility σ, then by using Itô formula [9], we can prove that

$$Y_t = Y_0X_t^k e^{-0.5k(k-1)\sigma^2 t}.$$

Hence we can conclude that compounding leveraged trading is approximately the same as trading with power perpetuals. Readers find in Section 4.5.2 for further discussion on compounding capital efficiency and leverage. In the following section, we shall utilize power functions to construct pay-off functions for perpetual future market. However, due to the constraint of limited cash-flow, our goal is that, at any price of the indexing asset, the long value and short value are positively determined and bounded, avoiding infinite growth.

2.3 Power pay-off formulas

Definition 2.1. *Given $\alpha, \beta > 0$, and $k > 0, R > 0$, we define a pair of dual pay-off functions:*

The long pay-off value is

$$\Phi(x) = \Phi(k, x) = \begin{cases} \alpha x^k & \textit{if} \quad \alpha x^k \le \frac{R}{2} \quad \textit{i.e} \quad x \le \left(\frac{R}{2\alpha}\right)^{1/k} \\ R - \frac{R^2}{4\alpha x^k} & \textit{otherwise} \quad \textit{i.e} \quad x > \left(\frac{R}{2\alpha}\right)^{1/k} \end{cases} \tag{1}$$

The short pay-off value is

$$\Psi(x) = \Phi(-k, x) = \begin{cases} \beta x^{-k} & \textit{if} \quad \beta x^{-k} \le \frac{R}{2} \quad \textit{i.e} \quad x \ge \left(\frac{2\beta}{R}\right)^{1/k} \\ R - \frac{R^2}{4\beta x^{-k}} = R - \frac{R^2 x^k}{4\beta} & \textit{otherwise} \quad \textit{i.e} \quad x < \left(\frac{2\beta}{R}\right)^{1/k} \end{cases} \tag{2}$$

We observe that Φ and Ψ are a concatenation of a power function and its inverse version (see Fig. 1 for an illustration). On the defined domain, Φ is strictly increasing, while Ψ is strictly decreasing. Moreover, for $m_\ell = \left(\frac{R}{2\alpha}\right)^{1/k}$ and $m_s = \left(\frac{2\beta}{R}\right)^{1/k}$, we have

$$\text{for} \quad 0 < x \le m_\ell, \quad \text{then} \quad \Phi(x) = \alpha x^k \le \frac{R}{2}, \qquad \lim_{x \to m_\ell^-} \Phi(x) = \lim_{x \to m_\ell^-} \alpha x^k = \frac{R}{2},$$

$$\text{for} \quad x > m_\ell \quad \text{then} \quad \Phi(x) = R - \frac{R^2}{4\alpha x^k} \ge \frac{R}{2}, \qquad \lim_{x \to m_\ell^+} \Phi(x) = \lim_{x \to m_\ell^+} \left(R - \frac{R^2}{4\alpha x^k}\right) = \frac{R}{2},$$

and

$$\text{for} \quad x \ge m_s, \quad \text{then} \quad \Psi(x) = \beta x^{-k} \le \frac{R}{2}, \qquad \lim_{x \to m_s^+} \Phi(x) = \lim_{x \to m_s^+} \beta x^k = \frac{R}{2},$$

$$\text{for} \quad 0 < x < m_s \quad \text{then} \quad \Psi(x) = R - \frac{R^2 x^k}{4\beta} \ge \frac{R}{2}, \qquad \lim_{x \to m_s^-} \Psi(x) = \lim_{x \to m_s^-} \left(R - \frac{R^2 x^k}{4\beta}\right) = \frac{R}{2}.$$

This implies that Φ, Ψ are continuous on $(0; +\infty)$. Further, we will prove that they are differentiable on their domains, in particular, at $m_\ell = \left(\frac{R}{2\alpha}\right)^{1/k}$ and $m_s = \left(\frac{2\beta}{R}\right)^{1/k}$, respectively.

$$\lim_{x \to m_\ell^-} \frac{\mathrm{d}\Phi}{\mathrm{dx}} = \alpha k x^{k-1}|_{m_\ell^-} = \frac{kR}{2}\left(\frac{2\alpha}{R}\right)^{1/k}, \qquad \text{and} \quad \lim_{x \to m_\ell^+} \frac{\mathrm{d}\Phi}{\mathrm{dx}} = \frac{kR^2}{4\alpha x^{k+1}}|_{m_\ell^+} = \frac{kR}{2}\left(\frac{2\alpha}{R}\right)^{1/k};$$

and

$$\lim_{x \to m_s^-} \frac{\mathrm{d}\Psi}{\mathrm{dx}} = -\beta k x^{-k-1}|_{m_s^-} = \frac{-kR}{2}\left(\frac{R}{2\beta}\right)^{1/k}, \qquad \text{and} \quad \lim_{x \to m_s^+} \frac{\mathrm{d}\Psi}{\mathrm{dx}} = \frac{-kR^2}{4\beta} x^{k-1}|_{m_s^+} = \frac{-kR}{2}\left(\frac{R}{2\beta}\right)^{1/k}.$$

Additionally, Φ, Ψ are bounded and asymptotic at infinity. Thus, we call Φ and Ψ ***asymptotic power dual curves*** or for short ***asymptotic power curves*** (see examples in Fig. 1).

$$\lim_{x \to +\infty} \Phi = \lim_{x \to +\infty} \left(R - \frac{R^2}{4\alpha x^k}\right) = R, \qquad \lim_{x \to +\infty} \Psi = \lim_{x \to +\infty} \beta x^{-k} = 0.$$

The intersecting point (e.g. m_ℓ, m_s) of the two branches made up of each asymptotic curve is also the inflection point of the curve, presenting the curvature change from convex to concave (see Fig. 1). The inflection points of the two curves are the same if and only if $4\alpha\beta = R^2$. Theorem 2.2 is clear by the previous observations.

Theorem 2.2. *Given $\alpha, \beta > 0$, and $k > 0, R > 0$, we have:*

- *Φ and Ψ are continuous and differentiable on the domain $(0; +\infty)$;*
- *Φ and Ψ are asymptotic at infinity and bounded within $(0; R)$.*

Theorem 2.3. *For $\alpha > 0, \beta > 0$, and $R > 0$, for some $x = x_0$, if it holds true*

$$\Phi(x) + \Psi(x) \leq R \tag{3}$$

then the inequality is true for all $x \in (0; \infty)$. Moreover, equality happens if and only if $4\alpha\beta = R^2$, i.e. the two asymptotic curves are symmetric via the horizontal line $y = R/2$.

Proof. According to Eq. (1) and Eq. (2), and the inequality (3) holds true for some $x = x_0 < \min\left\{\left(\frac{R}{2\alpha}\right)^{1/k}, \left(\frac{2\beta}{R}\right)^{1/k}\right\}$. We will prove that the inequality is true for all $x \in (0; \infty)$.

For $x = x_0 < \min\left\{\left(\frac{R}{2\alpha}\right)^{1/k}, \left(\frac{2\beta}{R}\right)^{1/k}\right\}$, because the inequality (3) holds true for some $x = x_0$,

$$\Phi(x_0) + \Psi(x_0) = \alpha x_0^k + R - \frac{R^2 x_0^k}{4\beta} = R + \left(\alpha - \frac{R^2}{4\beta}\right) x_0^k \leq R,$$

$$\text{it implies} \quad \left(\alpha - \frac{R^2}{4\beta}\right) x_0^k \leq 0 \Leftrightarrow \alpha - \frac{R^2}{4\beta} \leq 0 \Leftrightarrow \beta \leq \frac{R^2}{4\alpha} \Leftrightarrow \frac{2\beta}{R} \leq \frac{R}{2\alpha}. \tag{4}$$

This, in turn, makes the inequality (3) holds true for all $0 < x < \min\left\{\left(\frac{R}{2\alpha}\right)^{1/k}, \left(\frac{2\beta}{R}\right)^{1/k}\right\} = \left(\frac{2\beta}{R}\right)^{1/k}$, i.e.

$$\Phi(x) + \Psi(x) = \alpha x^k + R - \frac{R^2 x^k}{4\beta} = R + \left(\alpha - \frac{R^2}{4\beta}\right) x^k \leq R. \tag{5}$$

By (4), for $\left(\frac{2\beta}{R}\right)^{1/k} \leq x \leq \left(\frac{R}{2\alpha}\right)^{1/k}$, we have $\frac{2\beta}{R} \leq x^k \leq \frac{R}{2\alpha}$, and it holds

$$\Phi(x) + \Psi(x) = \alpha x^k + \beta x^{-k} \leq \frac{R}{2} + \frac{R}{2} = R.$$

For $x > \left(\frac{R}{2\alpha}\right)^{1/k}$, by (4), it holds true $\beta - \frac{R^2}{4\alpha} \leq 0$ and

$$\Phi(x) + \Psi(x) = R - \frac{R^2}{4\alpha x^k} + \beta x^{-k} = R + \left(\beta - \frac{R^2}{4\alpha}\right) x^{-k} \leq R. \tag{6}$$

When $4\alpha\beta = R^2 \Leftrightarrow \alpha = \frac{R^2}{4\beta} \Leftrightarrow \frac{R}{2\alpha} = \frac{2\beta}{R}$, it is clear that equality in (5) and (6) happens. This means $\Phi(x) + \Psi(x) = R$ for all x. The proof is now complete. □

We re-state Theorem 2.3 and have two corollaries as follows.

Theorem 2.4. *For $\alpha > 0, \beta > 0, R > 0$, if this inequality condition is satisfied*

$$4\alpha\beta \leq R^2, \tag{7}$$

then it holds true

$$\Phi(x) + \Psi(x) \leq R$$

for all $x \in (0; \infty)$. Moreover, equality happens exactly if and only if $4\alpha\beta = R^2$, i.e. the two asymptotic curves are symmetric via the horizontal line $y = R/2$.

Corollary 2.4.1. *For $\alpha > 0, \beta > 0, R > 0$, and $4\alpha\beta = R^2$, then it holds true*

- *$\Phi(x) + \Psi(x) = R$ for all x;*
- *The inflection points of the two curves coincide at $(\frac{R}{2\alpha}, \frac{R}{2})$;*
- *Two curves are symmetric via the horizontal line $y = R/2$.*

Corollary 2.4.2. *For $\alpha > 0, \beta > 0, R > 0$, and $4\alpha\beta < R^2$, then we have strictly inequality for all x*

$$\Phi(x) + \Psi(x) < R.$$

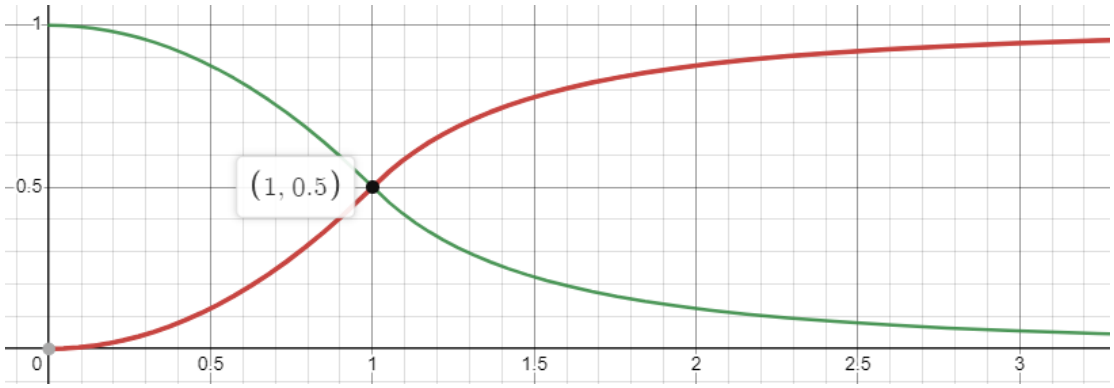


Fig. 1. An example of two (long-short) asymptotic power curves, for $R = 1, k = 2, \alpha = \beta = 0.5$. They are symmetric via the horizontal line $y = 0.5$ with the same inflection point $(1, 0.5)$.

3 A Novel Model for Derivatives Market

We will leverage the asymptotic power curves Φ and Ψ defined in Definition 2.1 as the long-short payoff functions to establish a derivatives market. Theorem 2.2 guarantees that long and short values are well-determined and bounded, while Theorem 2.3 and Theorem 2.4 ensure that the (long-short) market always operates normally after properly initiated despite any extremely severe cases. Formally, we define a decentralized derivatives market and payoff functions as follows.

Definition 3.1. *A decentralized derivatives market (or pool) is a tuple $\mathcal{M}(R, \alpha, \beta, \Phi(x), \Psi(x))$ defined by its liquidity pool (R, α, β) and the pair of long-short pay-off values $(\Phi(x), \Psi(x))$. The following conditions are to determine a valid market state, including its (first) initial state, or any state.*

Liquidity pool (R, α, β):

- *Initiate* ***reserve*** $R > 0$;
- *Initiate* ***long coefficient*** $\alpha > 0$;
- *Initiate* ***short coefficient*** $\beta > 0$;
- *It must hold* $4\alpha\beta \leq R^2$.

Setting the dual pair of payoff functions $(\Phi(x), \Psi(x))$ for power perpetuals:

- *Initiate* ***power leverage coefficient*** $1 < k \in \mathbb{Z}$
- *Initiate a price oracle for indexing price $x > 0$ of the underlying asset.*
- *Initiate* ***long pay-off*** *function* $\Phi(x) = \Phi(k, R, \alpha, x) = \begin{cases} \alpha x^k & \text{if} \quad x \leq \left(\frac{R}{2\alpha}\right)^{1/k} \\ R - \frac{R^2}{4\alpha x^k} & \text{otherwise} \quad \text{i.e} \quad x > \left(\frac{R}{2\alpha}\right)^{1/k} \end{cases}$
- *Initiate* ***short pay-off*** *function* $\Psi(x) = \Phi(-k, R, \beta, x) = \begin{cases} \beta x^{-k} & \text{if} \quad x \geq \left(\frac{2\beta}{R}\right)^{1/k} \\ R - \frac{R^2 x^k}{4\beta} & \text{otherwise} \quad \text{i.e} \quad x < \left(\frac{2\beta}{R}\right)^{1/k} \end{cases}$

We call $\Phi(x), \Psi(x)$ a dual pair because $\Psi(x) := \Psi(k, R, \beta, x) = \Phi(-k, R, \beta, x)$. A long (or short) position is determined by an amount of long (or short) tokens received from Derivable's markets. In the next, we shall compute state transitions of a pool, pricing of power perpetual positions and determine their optimal profit strategies.

3.1 Market state transition

In this section, we shall clarify the state transition of the defined market $\mathcal{M}$ in Definition 3.1, which is computed whenever one of the requests (add/remove liquidity, open/close long, open/close short) is executed.

Definition 3.2. *At an arbitrary price x, the transition of the power perpetuals pool $\mathcal{M}(R, \alpha, \beta, \Phi(x), \Psi(x))$ to a new state $\mathcal{M}_t(R_t, \alpha_t, \beta_t, \Phi_t(x), \Psi_t(x))$ corresponding to a request execution are defined as follows (for a changing amount $0 < \Delta R, \Delta R_L, \Delta R_S$).*

$$\text{Add liquidity:} \quad R + \Delta R \to R_t, \quad \Phi(x) \to \Phi_t(x), \quad \Psi(x) \to \Psi_t(x) \tag{8}$$
$$\text{Remove liquidity:} \quad R - \Delta R \to R_t, \quad \Phi(x) \to \Phi_t(x), \quad \Psi(x) \to \Psi_t(x) \tag{9}$$
$$\text{Open long:} \quad R + \Delta R_L \to R_t, \quad \Phi(x) + \Delta R_L \to \Phi_t(x), \quad \Psi(x) \to \Psi_t(x) \tag{10}$$
$$\text{Close long:} \quad R - \Delta R_L \to R_t, \quad \Phi(x) - \Delta R_L \to \Phi_t(x), \quad \Psi(x) \to \Psi_t(x) \tag{11}$$
$$\text{Open short:} \quad R + \Delta R_S \to R_t, \quad \Phi(x) \to \Phi_t(x), \quad \Psi(x) + \Delta R_S \to \Psi_t(x) \tag{12}$$
$$\text{Close short:} \quad R - \Delta R_S \to R_t, \quad \Phi(x) \to \Phi_t(x), \quad \Psi(x) - \Delta R_S \to \Psi_t(x) \tag{13}$$

By Definition 3.1, we must compute other quantities in the market tuple $\mathcal{M}$, given equations (8-13). We compute α_t, β_t from the previous long-short values (i.e. old states) $\Phi(x), \Psi(x)$, respectively, as follows.

- *Add liquidity* $(+\Delta R)$ and *remove liquidity* $(-\Delta R)$, from equations (8, 9),

  We compute

  $$R_t = R \pm \Delta R;$$
  $$\alpha_t = \begin{cases} \frac{\Phi(x)}{x^k} = \alpha & \text{if} \quad x \le \left(\frac{R_t}{2\alpha_t}\right)^{1/k} \\ \frac{R_t^2}{4(R_t - \Phi(x))x^k} & \text{otherwise} \end{cases}; \qquad \beta_t = \begin{cases} \frac{\Psi(x)}{x^{-k}} = \beta & \text{if} \quad x \ge \left(\frac{2\beta_t}{R_t}\right)^{1/k} \\ \frac{R_t^2 x^k}{4(R_t - \Psi(x))} & \text{otherwise} \end{cases};$$

  Re-check $4\alpha_t\beta_t \le R_t^2$ holding true;
  Re-check $\Phi_t(R_t, \alpha_t, x) = \Phi(R, \alpha, x)$ holding true;
  Re-check $\Psi_t(R_t, \beta_t, x) = \Psi(R, \beta, x)$ holding true.

- *Open long* $(+\Delta R_L)$ and *close long* $(-\Delta R_L)$, from equations (10, 11),

  We compute

  $$R_t = R \pm \Delta R_L;$$
  $$\alpha_t = \begin{cases} \frac{\Phi(x) \pm \Delta R_L}{x^k} & \text{if} \quad x \le \left(\frac{R_t}{2\alpha_t}\right)^{1/k} \\ \frac{R_t^2}{4(R_t - (\Phi(x) \pm \Delta R_L))x^k} & \text{otherwise} \end{cases}; \qquad \beta_t = \begin{cases} \frac{\Psi(x)}{x^{-k}} = \beta & \text{if} \quad x \ge \left(\frac{2\beta_t}{R_t}\right)^{1/k} \\ \frac{R_t^2 x^k}{4(R_t - \Psi(x))} & \text{otherwise} \end{cases};$$

  Re-check $4\alpha_t\beta_t \le R_t^2$ holding true;
  Re-check $\Phi_t(R_t, \alpha_t, x) = \Phi(R, \alpha, x) \pm \Delta R_L$ holding true;
  Re-check $\Psi_t(R_t, \beta_t, x) = \Psi(R, \beta, x)$ holding true.

- *Open short* $(+\Delta R_S)$ and *close short* $(-\Delta R_S)$, from equations (12, 13), we compute

  $$R_t = R \pm \Delta R_S;$$
  $$\alpha_t = \begin{cases} \frac{\Phi(x)}{x^k} = \alpha & \text{if} \quad x \le \left(\frac{R_t}{2\alpha_t}\right)^{1/k} \\ \frac{R_t^2}{4(R_t - \Phi(x))x^k} & \text{otherwise} \end{cases}; \qquad \beta_t = \begin{cases} \frac{\Psi(x) \pm \Delta R_S}{x^{-k}} & \text{if} \quad x \ge \left(\frac{2\beta_t}{R_t}\right)^{1/k} \\ \frac{R_t^2 x^k}{4(R_t - (\Psi(x) \pm \Delta R_S))} & \text{otherwise} \end{cases};$$

  Re-check $4\alpha_t\beta_t \le R_t^2$ holding true;
  Re-check $\Phi_t(R_t, \alpha_t, x) = \Phi(R, \alpha, x)$ holding true;
  Re-check $\Psi_t(R_t, \beta_t, x) = \Psi(R, \beta, x) \pm \Delta R_S$ holding true.

3.2 Remark

According to Condition (7) in Theorem 2.4, in order for the new market state $\mathcal{M}_t$ to be valid, we must check

$$4\alpha_t\beta_t \leq R_t^2$$

holds true for all transition cases. Then, we obtain a new dual pair of asymptotic curves (corresponding with R_t, α_t, β_t) to compute the next long-short values in the following market state transition. The new long-short curves are:

$$\Phi_t(x) = \Phi_t(R_t, \alpha_t, x) = \begin{cases} \alpha_t x^k & \text{if} \quad x \leq \left(\frac{R_t}{2\alpha_t}\right)^{1/k} \\ R_t - \frac{R_t^2}{4\alpha_t x^k} & \text{otherwise} \end{cases};$$

$$\Psi_t(x) = \Psi_t(R_t, \beta_t, x) = \begin{cases} \beta_t x^{-k} & \text{if} \quad x \geq \left(\frac{2\beta_t}{R_t}\right)^{1/k} \\ R_t - \frac{R_t^2 x^k}{4\beta_t} & \text{otherwise} \end{cases}.$$

We completely introduce a novel model for power perpetual derivatives market based on the invention of asymptotic power curves. In Section 4, we will analyze characteristics of that model in terms of derivatives markets with those participants mostly concerned, for example, pricing method, leverage, decay factor, premium fee and impermanent loss. Note that, by mathematical nature, liquidation does not appear in our asymptotic power curve model.

3.3 Pricing long-short positions without decay factor

Pricing a derivatives is essential in finance. In this section, under Black-Schole model [3], we will apply perpetual American options framework (i.e. no expiration time) introduced by Nicolas Privault [10] to compute pricing of Derivable's power perpetuals without decay factor (which is introduced in Section 4.7). Assume that the price of the power-perpetuals token defined in Definition 2.1 follows a geometric Brownian motion.

$$\frac{dY_t}{Y_t} = \mu dt + \sigma dW_t.$$

The equation means that the growth rate of the token price is constant μ, perturbed by an aggregate Gaussian noise of size σ-also called return volatility.

In what follows, we assume that there is no incoming traders to the pool. The fair price of any pay-off stream $f(t, X_t)$, exercised by the buyer, at an optimal stopping time τ, is given by the net present value of the pay-off under the risk-neutral probability

$$V = \max_{\tau} \mathbf{E}^*[e^{-r\tau} f(\tau, X_\tau)].$$

What is risk-neutral probability? It is an adjustment of the statistical probability under which the dynamics of token price is

$$\frac{dX_t}{X_t} = rdt + \sigma dW_t.$$

Why μ is replaced by r? Because r is the funding rate and the whole pricing methodology depends only on the hedging arguments, which relates constantly buying and selling tokens (thus the growth rate term will be removed, while cash is borrowed and lent at interest rate r).

Without loss of generality, assume that $X_0 = 1$.

3.3.1 Pricing long positions

Let's denote

$$a = \left[\frac{R}{2\alpha}\right]^{1/k}, C_k = \left[\frac{k+1}{2}\right]^{1/k}, L^* = aC_k.$$

The optimal stopping time is defined by the first passage time of the token price cross a threshold $L > 1$ that we have to determine.

$$\tau = \inf\{t > 0 : X_t \geq L\}.$$

It is straightforward to prove that

$$\mathbf{E}^*[e^{-r\tau}] = \frac{X_0}{L} = \frac{1}{L}.$$

Recall that if $a > 1$, the initial state $(X_0, Q(X_0))$, where $Q(x) = \Phi(x)/\alpha$ belongs to the convex branch on the left, otherwise it belongs to the concave branch on the right.

We consider two cases.

Case 1: $L^* > 1$. This case includes the case where the initial states belongs to the left branch ($a > 1$) and a part of the right branch. We can prove that the optimal threshold is given by L^*. And the optimal pay-off value is

$$Q^* = Q(L^*) = 2a^k - a^{2k}[L^*]^{-k} = \frac{2k}{k+1}a^k.$$

The price of the contract is

$$V^* = \mathbf{E}^*[e^{-r\tau^*}Q^*] = \frac{Q^*}{L^*} = \frac{4k}{(k+1)^2}(L^*)^k > 1.$$

The optimal strategy is to wait until the state is a bit beyond the inflection point to the right. For example, if $k = 2, L^* = 1.1$ (i.e. traders have to wait until the price increases by 10% to exercise), then the price is 1.0755. This means that the funding rate is about 7.55%.

Case 2: $L^* < 1$ (the tail of the concave branch). We can prove that it is optimal to execute the position right at the initial time $t = 0$. This means that no trader is interested in open long position at the right tail of the pay-off curve.

3.3.2 Pricing short positions

Let's denote

$$b = \left[\frac{2\beta}{R}\right]^{1/k}, c = \frac{2r}{\sigma^2} < 1, C_{-k} = \left[\frac{2c}{c+k}\right]^{1/k}, L_* = bC_{-k}.$$

Denote the stopping time

$$\tau = \inf\{t > 0 : X_t \leq L\}.$$

It is straightforward to prove that

$$\mathbf{E}^*[e^{-r\tau}] = \left[\frac{X_0}{L}\right]^{-c} = L^c < 1.$$

Recall that if $b < 1$, the initial state $(X_0, Q(X_0))$, where $Q(x) = \Psi(x)/\beta$ belongs to the convex branch on the right, otherwise it belongs to the concave branch on the left.

We consider two cases.

Case 1. $L_* < 1$. This case includes the case where the initial states belongs to the right branch ($b < 1$) and a part of the left branch. We can prove that the optimal threshold is given by L_*. And the optimal pay-off value is

$$Q^* = Q(L_*) = \frac{2}{b^k} - \frac{1}{b^{2k}}[L_*]^k = \frac{2k}{k+c}b^{-k}.$$

The price of the contract is

$$V^* = \mathbf{E}^*[e^{-r\tau^*}Q^*] = Q^*L_*^c = \frac{2k}{(k+c)C_{-k}}(L_*)^{-\frac{k}{c}} > 1.$$

The optimal strategy is to wait until the state is a bit beyond the inflection point to the left.

Case 2. $L_* > 1$ (the head of the concave branch) We can prove that it is optimal to execute the position right at the initial time $t = 0$. This means that no trader is interested in open short position at the left head of the pay-off curve.

3.3.3 Interactive behaviours of traders in a trading pool

From the above analysis, there are three typical cases

- If $L^* < 1$: there are only traders who open Long positions.
- If $L_* > 1$: there are only traders who open Short positions.
- $L_* < 1 < L^*$: both Long and Short are traded. Whenever the token price moves such that one of the two boundary conditions happens with X_t^k replacing 1, then one of the two side (Long or Short) will execute and exit the game.

4 Model Analysis

4.1 Liquidity reserve vs counterparty liquidity

Counterparty in traditional finance means the other party that participates in a financial transaction. In a narrow sense, counterparties are market makers who provide liquidity and facilitate trades. In DeFi, in particular, regarding Derivable markets, counterparties are market makers and liquidity providers (LPs), who provide liquidity for long and short traders. Market makers are creators (or initial LPs) of certain power perpetual markets, while LPs follow market makers to add liquidity to the created pools. Thus, LPs are less exposed to price volatility but to impermanent liquidity loss and gain (see Section 4.3). Roughly speaking, **counterparty liquidity** is provided by liquidity providers.

Note that all **liquidity reserve** (for short, reserve) of the market is R, including counterparty liquidity and funds deposited by long-short traders. By Theorem 2.4, total unresolved profit & loss (PnL) is $\Phi(x)+\Psi(x)$ always less or equal to R for all x. Formally, the quantity $\mathcal{L}(x) = R - \Phi(x) - \Psi(x) \geq 0$ is called **counterparty liquidity** of the market, provided by liquidity providers. Greater counterparty liquidity $\mathcal{L}(x)$ means greater liquidity reserve R, and greater liquidity reserve allows a wider price range with full leverage for traders, and vice versa. According to Equations (8, 9), (counterparty) liquidity is added or removed from the market $\mathcal{M}$ without affecting long-short values. In fact, our market model can operate normally without counterparty liquidity, i.e. only long and short traders appear in the market. However, liquidity providers should be incentivized to provide more counterparty liquidity $\mathcal{L}$, hence increasing efficiency and leverage zoom for long-short traders (see Section 4.7 for funding rate incentives).

By Definition 2.1, it is clear that optimal leverage almost lies on the left side of the inflection points, i.e. power branches of the asymptotic power curves. However, the two inflection points of the two dual long-short curves may not be the same. We will find the maximum value of counterparty liquidity for each market.

Recall that the long curve Φ is strictly increasing from 0 to R, while the short curve Ψ is strictly decreasing from R to 0. Thus, they must intersect exactly at a unique point. By Corollary 2.4.1, if $4\alpha\beta = R^2$, then $\mathcal{L} = R - \Phi(x) - \Psi(x) = 0$ for all x. However, in practice, we expect a positive counterparty liquidity $\mathcal{L}$. Therefore, by Corollary 2.4.2, we only consider $4\alpha\beta < R^2 \Leftrightarrow \frac{2\beta}{R} < \frac{R}{2\alpha}$ then $\mathcal{L}(x) = R - \Phi(x) - \Psi(x) > 0$ for all x. For $\left(\frac{2\beta}{R}\right)^{1/k} \leq x \leq \left(\frac{R}{2\alpha}\right)^{1/k}$, we have

$$\Phi(x) = \Psi(x) \Leftrightarrow \alpha x^k = \beta x^{-k} \Leftrightarrow x = \left(\frac{\beta}{\alpha}\right)^{\frac{1}{2k}} \quad \text{(satisfied).}$$

We will show that the unique intersecting point $x = \left(\frac{\beta}{\alpha}\right)^{\frac{1}{2k}}$ of the two long-short curves is also the maximum point of liquidity, given $4\alpha\beta < R^2$.

$$\mathcal{L}(x) = R - \Phi(x) - \Psi(x) = \begin{cases} R - \alpha x^k - (R - \frac{R^2 x^k}{4\beta}) = (\frac{R^2}{4\beta} - \alpha)x^k & \text{if} \quad x < \left(\frac{2\beta}{R}\right)^{1/k} \\ R - \alpha x^k - \beta x^{-k} & \text{if} \quad \left(\frac{2\beta}{R}\right)^{1/k} \leq x \leq \left(\frac{R}{2\alpha}\right)^{1/k} \\ R - (R - \frac{R^2}{4\alpha x^k}) - \beta x^{-k} = (\frac{R^2}{4\alpha} - \beta)x^{-k} & \text{otherwise} \end{cases} \tag{14}$$

Because function $\mathcal{L}(x)$ is increasing at the first interval and decreasing at the third one. At least one critical point of $L(x)$ must belong to the second (between) interval. We have

$$\mathcal{L}'(x) = -k\alpha x^{k-1} + k\beta x^{-k-1} \qquad \mathcal{L}'(x) = 0 \Leftrightarrow x = \left(\frac{\beta}{\alpha}\right)^{\frac{1}{2k}}$$

$$\mathcal{L}'(x) > 0 \Leftrightarrow x < \left(\frac{\beta}{\alpha}\right)^{\frac{1}{2k}} \qquad \mathcal{L}'(x) < 0 \Leftrightarrow x > \left(\frac{\beta}{\alpha}\right)^{\frac{1}{2k}}.$$

Thus, $x = \left(\frac{\beta}{\alpha}\right)^{\frac{1}{2k}}$ is the maximum point of liquidity function $\mathcal{L}(x)$. Additionally, the function is asymptotic at infinity (see Fig. 2 for an illustration).

$$\lim_{x\to\infty} \mathcal{L}(x) = 0.$$

From Eq. (14), we have $\Phi(x) + \mathcal{L}(x) = R - \Psi(x)$. This equation demonstrates an easily comprehensive view at the relation between long value, short value and counterparty liquidity (see Fig. 3). In the next sections, sometimes, we use $R - \Psi(x)$ as an alternative of short value.

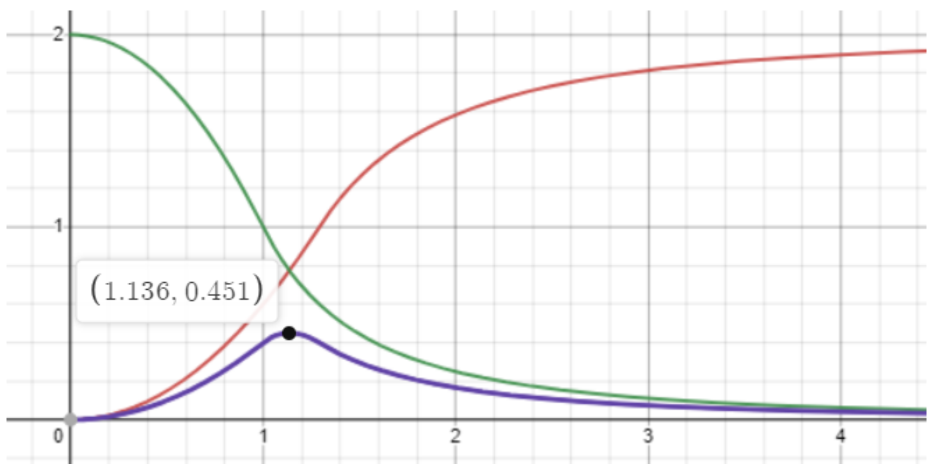


Fig. 2. Liquidity Elasticity phenomenon: long curve (red), short curve (green), counterparty liquidity curve (violet) with maximal point, for $k = 2, R = 2, \alpha = 0.6, \beta = 0.1$.

4.2 Entering a market

At the entry point, assuming a trader swaps an amount ΔR of fund for a long or short position of the market $\mathcal{M}(R, \alpha, \beta, \Phi(x), \Psi(x))$. He will concern about how much profit he can take from the current market. This is relative to available counterparty liquidity $\mathcal{L}(x) = R - \Phi(x) - \Psi(x)$. Greater counterparty liquidity $\mathcal{L}(x)$ allows optimal leverage and provides higher profit zoom, hence attracting more traders to enter and take profit upon their accurate price movement prediction. Vice versa, small liquidity $\mathcal{L}(x)$ discourages traders from entering the market. However, counterparty liquidity has elasticity, up or down, as presented in Section 4.1 and Section 4.3, hence, traders can expect a larger liquidity zoom after their entry.

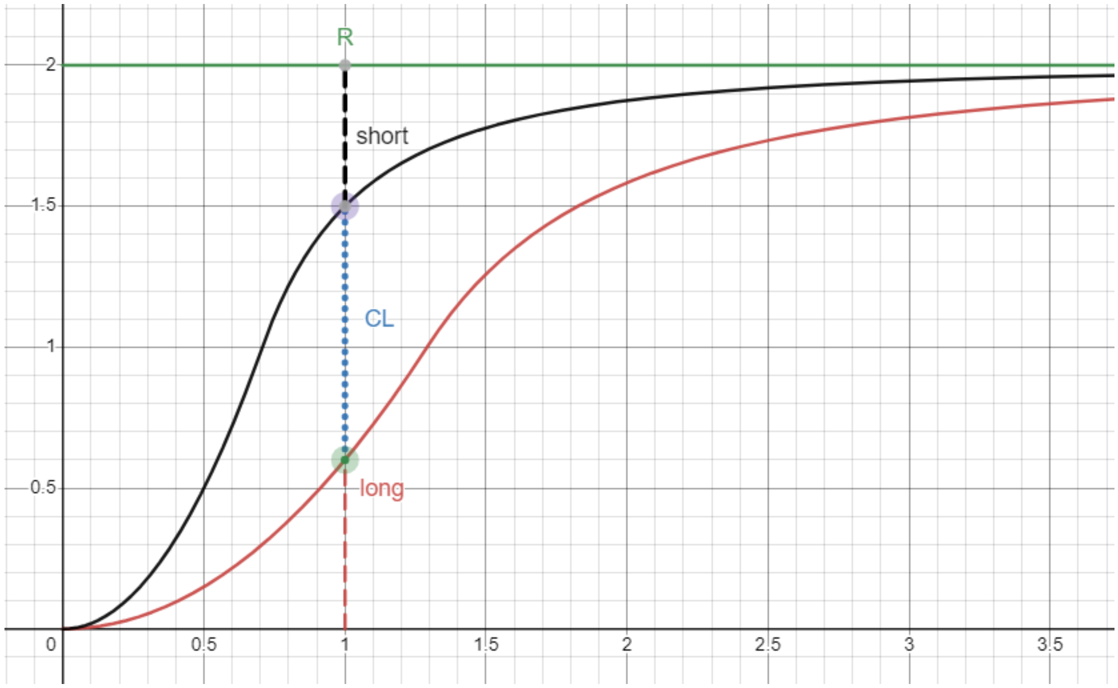


Fig. 3. Relationship between long-short values and counterparty liquidity: long curve (red) and long value (vertically red-dashed line), short curve (green) and short value (vertically black-dashed line), counterparty liquidity CL value (vertically blue-dotted line), for $k = 2, R = 2, \alpha = 0.6, \beta = 0.5$.

4.3 Liquidity elasticity: impermanent loss & gain

According to the previous mathematical analysis, counterparty liquidity $\mathcal{L}(x)$ (sometimes, for short saying "liquidity") is a smooth curve with a unique maximum and asymptotic behavior (see Fig. 2 for an illustration). More concretely, for a current market state, we calculate (counterparty) **liquidity elasticity** (LE) or liquidity PnL between two prices, from x to x_t, as follows.

$$
\begin{aligned}
LE &= \mathcal{L}(x_t) - \mathcal{L}(x) \\
LE &= (R - \Phi(x_t) - \Psi(x_t)) - (R - \Phi(x) - \Psi(x)) = (\Phi(x) + \Psi(x)) - (\Phi(x_t) + \Psi(x_t)) \\
&= (\Phi(x) - \Phi(x_t)) + (\Psi(x) - \Psi(x_t)).
\end{aligned}
$$

If $LE > 0$, liquidity value increases; otherwise, liquidity value decreases. Observing that on the left side of the maximum point, $LE > 0$, and on the right side of the maximum point, $LE < 0$ (see Fig. 2 also). More explicitly, we have the following theorem.

Theorem 4.1. *Given the market $\mathcal{M}(R, \alpha, \beta, \Phi(x), \Psi(x))$, and $x_t \neq x$, the following assertions holds for $LE = \mathcal{L}(x_t) - \mathcal{L}(x)$:*

- *If $x < x_t < \left(\frac{\beta}{\alpha}\right)^{\frac{1}{2k}}$, then $LE \times (x_t - x) > 0$ (i.e. varying in the same direction).*
- *If $\left(\frac{\beta}{\alpha}\right)^{\frac{1}{2k}} \leq x < x_t$, then $LE \times (x_t - x) < 0$ (i.e. varying in the opposite direction).*
- *If $\left(\frac{\beta}{\alpha}\right)^{\frac{1}{2k}}$ lies between x and x_t, then $LE = 0$ if and only if $\Phi(x) + \Psi(x) = \Phi(x_t) + \Psi(x_t)$ (i.e. total long and short values at the two price indexes are the same).*

In Uniswap's AMM model based on inverse function, liquidity providers usually suffer impermanent loss (IL) due to price volatility. IL phenomenon is a notable characteristic of Uniswap's AMM and the like. IL will become *permanent losses* if the liquidity is withdrawn before the price returns to the entry price. Our market model based on asymptotic power curves has the impermanent loss phenomenon, but also impermanent gain, which never appears in all other models. Formally, according to the indexing price change, we say:

- The **Impermanent loss** appears whenever liquidity elasticity $LE < 0$.
- The **Impermanent gain** appears whenever liquidity elasticity $LE > 0$.

4.4 Infinite liquidity

By Theorem 2.4 and Corollary 2.4.2, it is mathematically proven that whenever a valid market is initiated by condition (7), liquidity reserve is always non-negative, hence liquidity is always ensured. Section 4.3 shows the elasticity of liquidity in the market with asymptotic behavior at infinity, once again, re-confirms the infinity liquidity property of the Derivable model. It is shown and proved that the Derivable model is able to be everlasting for all circumstances, even severe market crashes or extremely high price volatility. However, liquidity providers should be incentivized to provide more counterparty liquidity $\mathcal{L}$, hence increasing efficiency and leverage zoom for long-short traders (see Section 4.7 for funding rate incentives).

4.5 Capital efficiency

In the financial market, investors are most concerned about capital efficiency, that is, the rate of profit or loss compared to entry capital. For spot trading, assuming that the entry price is x_0, for a new price x of the trading asset, spot capital efficiency is $CE_1 = \frac{x-x_0}{x_0}$. This computation type is non-compounding. Similarly to compound interest, for each term of computing efficiency, if the previous price and previous value are applied for the next round of calculation, then we have compound efficiency, e.g. spot compound capital efficiency is $CE_1 = \frac{x_{t+1}-x_t}{x_t}$, where x_t is the previous price, x_{t+1} is the current price. For other leveraged trading methods (e.g. futures and perpetual futures) with various pricing formulas, people usually compare with spot trading to measure capital efficiency. Formally, we define:

Definition 4.2. *Given an underlying asset with price $x > 0$, and a (derivative) pricing function $P(x) \neq x$.*

Assuming that the entry price is x_0, for any new price x of the underlying asset, non-compound capital efficiency is

$$CE = \frac{P(x) - P(x_0)}{P(x_0)}, \quad \textit{particularly for spot} \quad CE_1 = \frac{x - x_0}{x_0}.$$

For x_t is the current price, x_{t-1} is the previous price of the underlying asset, compound capital efficiency is

$$CE = \frac{P(x_t) - P(x_{t-1})}{P(x_{t-1})}, \quad \textit{particularly for compound spot} \quad CE_1 = \frac{x_t - x_{t-1}}{x_{t-1}}.$$

- *We call $L = \frac{CE}{CE_1}$ leverage rate, or leverage.*
- *We say leveraged efficiency if $|L| > 1$.*
- *We say deleveraged efficiency if $|L| =< 1$.*

In derivatives markets, legacy exchanges usually offer leveraged efficiency $|L| > 1$ to attract traders. We will study how leverage is computed in several scenarios. The following pricing functions and computation are simplified to understand the nature of capital efficiency and leverage. They may not completely fit actual pricing and computing methods in existing exchanges. We separate non-compound versus compound types to calculate efficiency.

4.5.1 Non-compound capital efficiency and leverage

Constant leverage: It reads for a linear pricing function $P(x) = x_0 + \gamma(x - x_0), \gamma \neq 0$, hence

$$CE = \frac{P(x) - P(x_0)}{P(x_0)} = \frac{\gamma(x - x_0)}{x_0} \quad \text{and} \quad L = \frac{CE}{CE_1} = \gamma.$$

This is a (simple) *linear capital efficiency* that induces *constant leverage* given non-compounding, similar to popular legacy perpetuals exchanges. If $\gamma = 1$, we have spot efficiency with leverage 1, and $\gamma > 0$ for long while $\gamma < 0$ for short.

Power leverage: It reads for a simple power-2 pricing function $P(x) = x^2$, hence

$$CE = \frac{x^2 - x_0^2}{x_0^2} \quad \text{and} \quad L = \frac{CE}{CE_1} = \frac{x^2 - x_0^2}{x_0^2}\frac{x_0}{x - x_0} = \frac{x + x_0}{x_0}.$$

This induces *power capital efficiency* and *power leverage*. Power leverage is not a constant (see Table 1).

Change rate	entry	1%	-1%	- 20%	20%	50%	-50%
Spot price	100	101	99	80	120	150	50
Linear long P_1	100	102	98	60	140	260	0
Capital efficiency		2%	-2%	-40%	40%	160%	-100%
Constant leverage		2	2	2	2	2	2
Linear short P_2	100	98	102	140	60	0	200
Linear cap efficiency		-2%	2%	40%	-40%	-100%	100%
Constant leverage		-2	-2	-2	-2	-2	-2
Power long P_3	10000	10201	9801	6400	14400	22500	2500
Power capital efficiency		2.01%	-1.99%	-36%	44%	125%	-75%
Power Leverage		2.01	-1.99	1.8	2.2	2.5	1.5
Power short P_4	0.0001	0.000098	0.000102	0.000156	0.0000694	0.0000444	0.0004
Power capital efficiency		-1.97%	-2.03%	56%	-31%	-56%	300%
Power leverage		-1.97	-2.03	-2.8	-1.5	-1.1	-6

Table 1
Non-compound efficiency for spot price, linear (long) pricing $P_1(x) = x_0 + 2(x - x_0)$, linear (short) pricing $P_2(x) = x_0 - 2(x - x_0)$, power (long) pricing $P_3(x) = x^2$ and (short) pricing $P_4(x) = x^{-2}$.

Change by period	1st entry	1%	-1%	- 20%	20%	50%	-50%
Compound spot	100	101	99.99	79.99	95.99	143.986	71.993
Linear long P_1	100	102	98.98	59.994	111.989	191.981	0
Cap efficiency		2%	-3%	-39%	87%	71%	-100%
Leverage		2	3	2	4	1.429	2.0
Linear short P_2	100	98	103.02	139.986	47.995	0	287.971
Cap efficiency		-2%	3%	40%	-52%	-100%	188%
Leverage		-2	-3	-2	-3	-2	-4
Power long P_3	10000	10201	9998	6398.7	9214.2	20731.9	5183.0
Cap efficiency		2.01%	-1.99%	-36%	44%	125%	-75%
Leverage		2.01	1.99	1.8	2.2	2.5	1.5
Power short P_4	0.0001	0.00009803	0.00010002	0.00015628	0.0001085	0.0000482	0.000193
Cap efficiency		-1.97%	2.03%	56%	-31%	-56%	300%
Leverage		-1.97	-2.03	-2.8	-1.5	-1.1	-6

Table 2
Compound efficiency for spot price, linear (long) pricing $P_1(x_t) = x_{t-1} + 2(x_t - x_{t-1})$, linear (short) pricing $P_2(x_t) = x_{t-1} - 2(x_t - x_{t-1})$, power (long) pricing $P_3(x_t) = x_t^2$, and (short) pricing $P_4(x_t) = x_t^{-2}$.

4.5.2 Compound capital efficiency and leverage

Compound leverage: In Section 2.2, we proved a financial meaning of power perpetuals as compounding leverage. For compounding efficiency, assuming that we use linear pricing and re-set a new entry for each time of computing capital efficiency and leverage. More concretely, assuming x_0 is the first entry price, and at the time of computing, the current price x_t will be set as the new entry price of the next computing. Mathematically, regarding linear pricing function, we have $P(x_t) = x_{t-1} + \gamma(x_t - x_{t-1}), \gamma \neq 0$, hence

$$CE = \frac{P(x_t) - P(x_{t-1})}{P(x_{t-1})} = \frac{x_{t-1} + \gamma(x_t - x_{t-1}) - (x_{t-2} + \gamma(x_{t-1} - x_{t-2}))}{x_{t-2} + \gamma(x_{t-1} - x_{t-2})}.$$

This computing method is similar to compounding interest; hence we call it *compound capital efficiency* and *compound leverage*. Compound leverage is not a constant (see Table 2).

4.6 Elasticity of power leverage

Leverage measurement is important to know how much traders exposure to profit and loss under the price volatility of underlying assets. It is also critical for systematic risk management. According to Definition 3.2, market state changes per successful execution (i.e. state transition). This transition is continuous according to long and short pay-off curves. Therefore, capital efficiency and leverage are compounded following the market state transition. Regarding the Derivable model, we are concerned with measuring how much the powered pay-off value changes when the price of the underlying asset moves $(\frac{x+\Delta x}{x} - 1) \times 100\% = \frac{\Delta x}{x} \times 100\%$. For easier investigation, we consider two cases: very small change and significant change.

4.6.1 Small price change: non-compound leverage

Formally, for a sufficiently **small price change** Δx of the indexing asset, we would like to know how much of long and short values (i.e. pay-off values) change, i.e. finding capital efficiency CE and power leverage (L) rate when the price changes from x to $x + \Delta x$. We calculate CE, L for long and short as follows (see Fig. 4).

$$
\begin{aligned}
CE_L &= \frac{\Phi(x+\Delta x)}{\Phi(x)} - 1 \\
L_L &= \left(\frac{\Phi(x+\Delta x)}{\Phi(x)} - 1\right)\frac{x}{\Delta x} = \frac{x}{\Phi(x)}\frac{\Phi(x+\Delta x)-\Phi(x)}{\Delta x} \qquad \text{and} \quad \lim_{\Delta x \to 0} L_L = \frac{x}{\Phi(x)}\Phi'(x). \\
CE_S &= \frac{\Psi(x+\Delta x)}{\Psi(x)} - 1 \\
L_S &= \left(\frac{\Psi(x+\Delta x)}{\Psi(x)} - 1\right)\frac{x}{\Delta x} = \frac{x}{\Psi(x)}\frac{\Psi(x+\Delta x)-\Psi(x)}{\Delta x} \qquad \text{and} \quad \lim_{\Delta x \to 0} L_S = \frac{x}{\Psi(x)}\Psi'(x).
\end{aligned}
$$

$$
\Phi(x) = \begin{cases} \alpha x^k & \text{if} \quad x \le \left(\frac{R}{2\alpha}\right)^{1/k} \\ R - \frac{R^2}{4\alpha x^k} & \text{otherwise} \end{cases} \qquad \Longrightarrow \qquad \Phi'(x) = \begin{cases} k\alpha x^{k-1} & \text{if} \quad x \le \left(\frac{R}{2\alpha}\right)^{1/k} \\ \frac{kR^2 x^{-k-1}}{4\alpha} & \text{otherwise} \end{cases}
$$

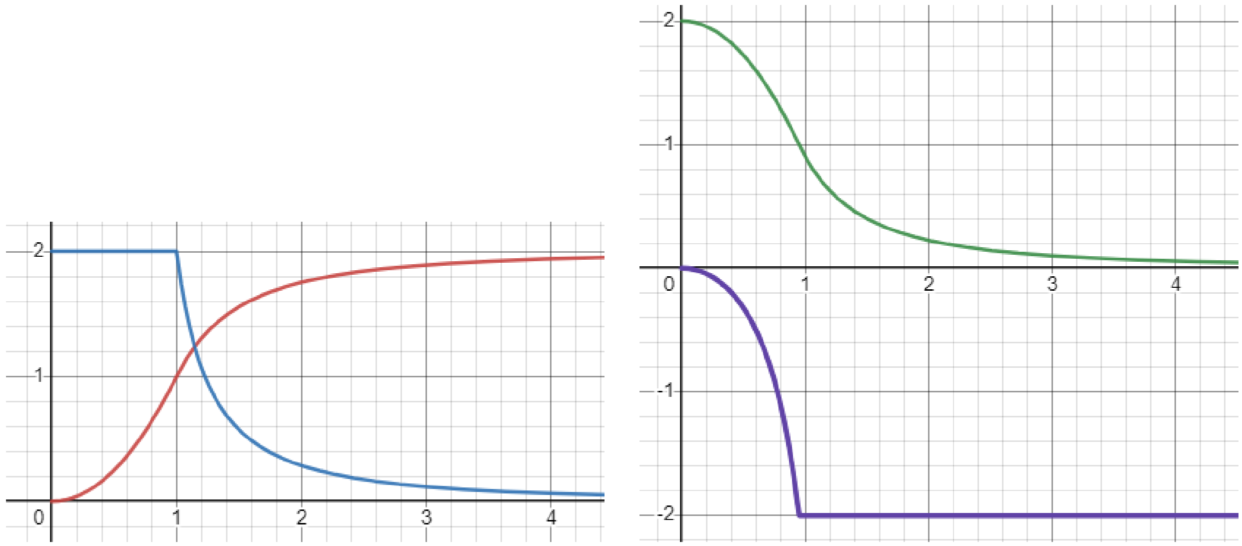
$$
\Psi(x) = \begin{cases} \beta x^{-k} & \text{if} \quad x \ge \left(\frac{2\beta}{R}\right)^{1/k} \\ R - \frac{R^2 x^k}{4\beta} & \text{otherwise} \end{cases} \qquad \Longrightarrow \qquad \Psi'(x) = \begin{cases} -k\beta x^{-k-1} & \text{if} \quad x \ge \left(\frac{2\beta}{R}\right)^{1/k} \\ -\frac{kR^2 x^{k-1}}{4\beta} & \text{otherwise} \end{cases}
$$

On the left side of the inflection points, i.e. power branches of the asymptotic power curves, we have:

$$
\begin{aligned}
L_L &= x\frac{\Phi'(x)}{\Phi(x)} = x\frac{k\alpha x^{k-1}}{\alpha x^k} = k; \\
L_S &= x\frac{\Psi'(x)}{\Psi(x)} = \frac{-kRx^k}{4\beta - Rx^k} = \frac{-kR + k\Psi(x)}{\Psi(x)} \quad \text{and} \quad \lim_{x \to 0} L_S = \lim_{\Psi(x) \to R} L_S = 0.
\end{aligned}
$$

On the right side of the inflection points, i.e. asymptotic branches of the long-short curves, we have:

$$
\begin{aligned}
L_L &= x\frac{\Phi'(x)}{\Phi(x)} = \frac{kR}{4\alpha x^k - R} = \frac{kR - k\Phi(x)}{\Phi(x)} \quad \text{and} \quad \lim_{x \to \infty} L_L = \lim_{\Phi(x) \to R} L_L = 0; \\
L_S &= x\frac{\Psi'(x)}{\Psi(x)} = -k.
\end{aligned}
$$



(a) Long leverage (blue) **(b)** Short leverage (violet)

Fig. 4. Leverage Elasticity phenomenon (very small price changes) with $k = 2, R = 2, \alpha = 1, \beta = 0.9$.
(**a**) Long curve (red), long leverage (blue segment) is 2 on the left of the inflection point (i.e. 2-time long leverage). On the right side, long leverage (blue curve) decreases with asymptote $y = 0$ when x goes to infinity.
(**b**) Short curve (green), short leverage (violet segment) is -2 on the right of the inflection point (i.e. 2-time short leverage). On the left side, short leverage (violet curve) increases to 0, varying in opposing direction with x.

4.6.2 Arbitrary price change: compound leverage

In general, for arbitrary price change $h \neq 0$ of the underlying asset, we shall investigate long-short compound capital efficiency CE and compound leverage L as functions of price x with parameter h. Compound leverage of the change between x and $x + h$ means that it is the summation of non-compound leverage associated with many small changes $h/N, N \in \mathbb{N}$, which is studied in Section 4.6.1. Formally, because of differentiability, we have

$$CE_L = \frac{\Phi(x+h)}{\Phi(x)} - 1 = \frac{1}{\Phi(x)} \int_x^{x+h} \Phi'(t)dt \qquad \text{and} \quad CE_S = \frac{\Psi(x+h)}{\Psi(x)} - 1 = \frac{1}{\Psi(x)} \int_x^{x+h} \Psi'(t)dt.$$

However, completely investigating the characteristics of compound leverage functions is rather hard. For practical purposes, we concern their behaviors on the left and right sides of the inflection points, i.e. on the left boundary at 0 and the right boundary at infinity (see Fig. 5). In the following, we utilize general forms of L'Hospital's rule to evaluate limits (of indeterminate-forms) of capital efficiency and leverage at boundary points.

On the left side of the inflection points, i.e. power branches of the asymptotic power curves, we have:

$$CE_L = \frac{\Phi(x+h)}{\Phi(x)} - 1 = \frac{\alpha(x+h)^k}{\alpha x^k} - 1 = \left(1 + \frac{h}{x}\right)^k - 1 \qquad \text{and} \quad \lim_{x \to 0} CE_L = +\infty.$$

$$L_L = \frac{CE_L}{CE_1} = \frac{(1+u)^k - 1}{u} \quad \text{for} \quad u = \frac{h}{x} \qquad \text{and} \quad \lim_{x \to 0} L_L = \lim_{u \to +\infty} k(1+u)^{k-1} = +\infty.$$

$$CE_S = \frac{\Psi(x+h)}{\Psi(x)} - 1 = \frac{Rx^k - R(x+h)^k}{4\beta - Rx^k} \qquad \text{and} \quad \lim_{x \to 0} CE_S = \frac{-Rh^k}{4\beta}.$$

$$L_S = \frac{CE_S}{CE_1} = \frac{Rx^k - R(x+h)^k}{4\beta - Rx^k} \times \frac{x}{h} \qquad \text{and} \quad \lim_{x \to 0} L_S = 0.$$

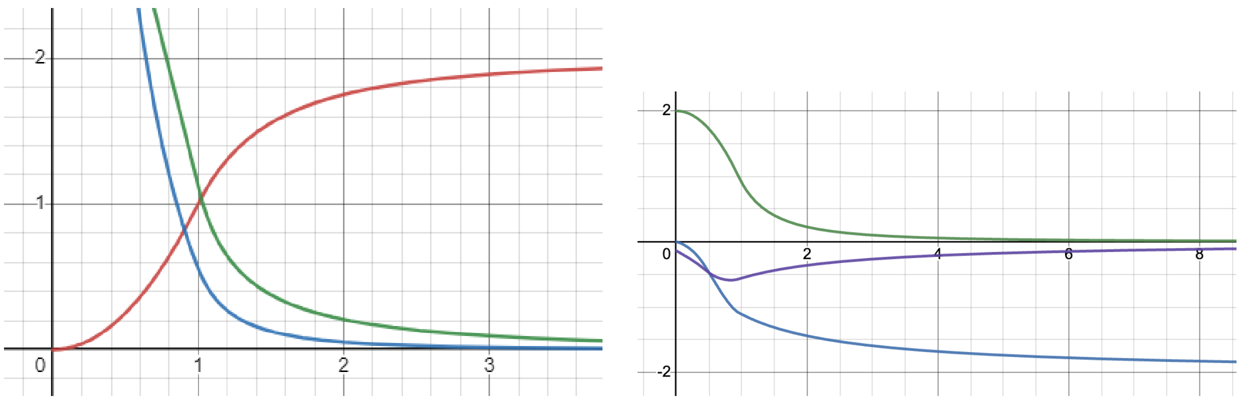
On the right side of the inflection points, i.e. asymptotic branches of the long-short curves, we have:

$$CE_L = \frac{\Phi(x+h)}{\Phi(x)} - 1 = \frac{R\left(1+\frac{h}{x}\right)^k - R}{\left(1+\frac{h}{x}\right)^k (4\alpha x^k - R)} \qquad \text{and} \quad \lim_{x\to+\infty} CE_L = 0.$$

$$L_L = \frac{CE_L}{CE_1} = \frac{R(1+u)^k - R}{(1+u)^k (4\alpha h^k u^{-k} - R)} \times \frac{1}{u} \quad \text{for} \quad u = \frac{h}{x} \qquad \text{and} \quad \lim_{x\to+\infty} L_L = \lim_{u\to 0} L_L = 0.$$

$$CE_S = \frac{\Psi(x+h)}{\Psi(x)} - 1 = \frac{x^k}{(x+h)^k} - 1 \qquad \text{and} \quad \lim_{x\to+\infty} CE_S(x) = 0.$$

$$L_S = \frac{CE_S}{CE_1} = \left(\frac{x^k}{(x+h)^k} - 1\right) \times \frac{x}{h} = \frac{1-(1+u)^k}{u(1+u)^k} \quad \text{for} \quad u = \frac{h}{x} \qquad \text{and} \quad \lim_{x\to+\infty} L_S = \lim_{u\to 0} L_S = -k.$$



(a) Long efficiency **(b)** Short efficiency

Fig. 5. Leverage Elasticity phenomenon (arbitrary changes) with $k = 2, R = 2, \alpha = 1, \beta = 0.9$, and $h = 0.5$
(**a**) Long curve (red), capital efficiency (blue) and leverage (green).
(**b**) Short curve (green), capital efficiency (violet) and leverage (blue).

Theorem 4.3. *For arbitrary price change h of the underlying asset, compound capital efficiency and compound leverage functions of price x with parameter h are convergent when x goes to infinity. Moreover, for any h, there exists a number M large enough such that compound leverage functions decrease on $x \in (M, +\infty)$.*

Long capital efficiency	$CE_L(x+h) = \dfrac{\Phi(x+h)}{\Phi(x)} - 1,$	$\lim_{x\to+\infty} CE_L(x+h) = 0.$
Long leverage	$L_L(x+h) = \left(\dfrac{\Phi(x+h)}{\Phi(x)} - 1\right) \times \dfrac{x}{h},$	$\lim_{x\to+\infty} L_L(x+h) = 0.$
Short capital efficiency	$CE_S(x+h) = \dfrac{\Psi(x+h)}{\Psi(x)} - 1,$	$\lim_{x\to+\infty} CE_S(x+h) = 0.$
Short leverage	$L_S(x+h) = \left(\dfrac{\Psi(x+h)}{\Psi(x)} - 1\right) \times \dfrac{x}{h},$	$\lim_{x\to+\infty} L_S(x+h) = -k.$

4.6.3 Summary on power leverage

Section 4.6 analyzes some properties and behaviors of compounding power capital efficiency and power leverage regarding long-short pricing curves. It shows a good match to our expectation, i.e. leverage should be adjusted automatically in accordance with market volatility. In fact, leverage management is very important to control system risk in all derivatives exchanges. Following the Derivable model, power perpetual markets derived from asymptotic power curves manage and adjust leverage continuously and correspondingly with price change and state transition, and the systematic risk is always well controlled. Whenever long value is greater than a haft of the liquidity reserve, its corresponding leverage starts to decrease (i.e. deleveraging), and similarly for short side. We provide Table 3 and Table 4 as examples of long-short capital efficiency and leverage on a Derivable's market.

Mathematical analysis in Sections 4.6.1, 4.6.2 and Theorem 4.3 shows that leverage of long-short sides is convergent when x goes to infinity (see Fig. 5), despite the change is small or large (i.e. for both compound and non-compound leverage).

Change rate	entry	1%	-1%	- 20%	20%	-95%	95%
Spot price	1	1.01	0.99	0.8	1.2	0.05	1.95
Square	1	1.0201	0.9801	0.64	1.44	0.0025	3.8025
Minus square	1	0.9803	1.0203	1.5625	0.6944	400	0.263
Power long value	1	1.02	0.9801	0.64	1.3116	0.0025	1.7517
Capital efficiency		2.01%	-1.99%	-36%	31%	-99.75%	75.17%
Leverage		2.01	1.99	1.8	1.56	1.05	0.79
Power short value	1	0.98	1.02	1.367	0.694	2.017	0.263
Capital efficiency		-1.97%	2.03%	37%	-31%	102%	-74%
Leverage		-1.97	-2.03	-1.8	-1.5	-1.1	-0.8

Table 3
Non-compound capital efficiency and leverage of long-short sides on Derivable market with $R = 2.02, k = 2, \alpha = \beta = 1$.

Change rate	entry	1%	-1%	- 20%	20%	-95%	95%
Spot price	1	1.01	0.9999	0.79992	0.959904	0.048	0.0936
Square	1	1.0201	0.9998	0.63987	0.92142	0.0023	0.00876
Minus square	1	0.9803	1.0002	1.56281	1.08529	434.1146	114.16557
Power long value	1	1.02	0.9998	0.63987	0.92142	0.0023	0.00876
Capital efficiency		2.01%	-1.98%	-36%	44%	-99.75%	280.25%
Leverage		2.01	1.98	1.8	2.2	1.05	2.95
Power short value	1	0.98	1	1.367	1.080	2.018	2.011
Capital efficiency		-1.97%	2.03%	37%	-21%	87%	-0.33%
Leverage		-1.97	-2.03	-1.8	-1.1	-0.9	-0.003

Table 4
Compound capital efficiency and leverage of long-short sides on Derivable market with $R = 2.02, k = 2, \alpha = \beta = 1$.

4.7 Continuous time-decay factor

It is clear that in legacy future markets, the funding rate is very important to balance market forces, hence keeping it in normal operation and closest reflecting spot price. However, computing it is difficult and complicated with many variables involved (visit computation of Binance's funding rate regarding perpetual future contracts). The funding rate also affects the fair-price of options or futures contracts, which is computed by the Black-Scholes model [3]. Almost centralized exchanges use the following funding rate formula:

```
Funding Rate = (Average Premium Index) + clamp(interest rate - Premium Index, 0.05%, -0.05%),
Funding Fee = (Nominal Value of Positions) x (Funding Rate),
Nominal Value of Positions = (Mark Price) x (Size of a Contract)
```

A decentralized perpetual exchange based on order-book, dYdX uses another formula

```
Funding Rate = (Premium Component / 8) + (Interest Rate Component),
Funding Fee = (Size of the Position) x (Index price by Oracle) x (Funding Rate)
```

For all existing perpetual exchanges based on unique positions, the funding rate is used for the long side and short side to compensate each other (then reducing the unbalancing gap between the two sides and, also the gap between perpetual future price and spot price). Interest rate plays an important role in all funding rate formulas which means traders borrowing fund to leverage their capital and position. Exchanges usually set interest rate of

0.01% daily, implying that a fund F approximately losses a half after 692 days (i.e. $F(1-0.001)^{692} \approx F/2$). This is similar to time-decay effect in option contracts.

Regarding Derivable model, we utilize time-decay factor (as continuous interest) and premium fee (see Section 4.9) to construct a suit of funding rate. The time-decay factor is to compensate liquidity providers (LPs), incentivizing them to provide liquidity in Derivable markets. Because of no liquidation in Derivable paradigm, traders pose much less risk compared to other future exchanges. This implies that Derivable's decay-rate is possibly higher than common loan to helps LPs reducing the risk of power-magnified impermanent loss. Liquidity providers earn decayed value from long and short positions, continuously.

According to mathematical analysis in Sections 4.6.1, 4.6.2 and Theorem 4.3, both compound and non-compound leverage of long-short markets is convergent when x goes to infinity, i.e. having the same behaviors if x large enough, whether the change is small or large. Therefore, we can define a continuous time-decay (similar as a flat interest rate) for Derivable's payoff function as follows.

Definition 4.4. *Given the market $\mathcal{M}(k, R, \alpha, \beta, \Phi(x), \Psi(x))$, we insert time-decay factor as a function of time $(0 < i = i(t) < 1)$ multiplying along with long-short pay-off functions as follows.*

- *Denote $H > 0$ is a half-life of decay, t is the variable of time, and $i = 2^{-t/H}$,*
- *Long value with time-decay* $\Phi(x) = \Phi(k, R, \alpha, i, x) = \begin{cases} i\alpha x^k & \text{if} \quad x \leq \left(\frac{R}{2\alpha}\right)^{1/k} \\ i\left(R - \frac{R^2}{4\alpha x^k}\right) & \text{otherwise} \quad \text{i.e} \quad x > \left(\frac{R}{2\alpha}\right)^{1/k} \end{cases}$
- *Short value with time-decay* $\Psi(x) = \Phi(-k, R, \beta, i, x) = \begin{cases} i\beta x^{-k} & \text{if} \quad x \geq \left(\frac{2\beta}{R}\right)^{1/k} \\ i\left(R - \frac{R^2 x^k}{4\beta}\right) & \text{otherwise} \quad \text{i.e} \quad x < \left(\frac{2\beta}{R}\right)^{1/k} \end{cases}$

With time-decay inserted into long-short payoff functions, conditions in Theorem 2.4, Corollary 2.4.2 and analysis in Section 4 are still valid to apply. The new parameterization of the market is $\mathcal{M}(k, R, i, \alpha, \beta, \Phi(x), \Psi(x))$. For a fixed price x, decay factor $i(t)$ implies a time-decayed long-short values and can be considered as a continuous, flat interest rate. At a half-life, we have $\Phi_{t=H}(x) = \frac{1}{2}\Phi_{t=0}(x)$, and $\Psi_{t=H}(x) = \frac{1}{2}\Psi_{t=0}(x)$. Consequently, the counterparty liquidity $R - \Phi_t(x) - \Psi_t(x)$ increases as t. Readers see Fig. 6 for an illustration on long-short values with time-decay.

4.8 What is the fair decay factor to cover IL for LPs?

Returning to long-short pricing in Section 3.3, we consider the case $L_* < 1 < L^*$, i.e. initially, both long and short participate in opening their positions. As the price of the positions is positive (i.e. traders have to pay the pool, $V > 1$), we want to determine the decay factor such that when this factor is applied on the holdings long positions α and short positions β, the price of the decayed positions will be equal to zero.

However, we can show that regardless of decay factor, the price of the positions is always positive. Why? It is straightforward to see that if the trader exercises the position right at the first time, then his gain is zero regardless of the decay factor. By continuity argument, for all decay factor, the price of the position should be positive. Therefore, the perpetual option pricing formula should be used instead of decay factor. However, due to dynamicity of the long-short pool, decay factor is still meaningful to incentivize liquidity provision.

4.9 Premium fee mechanism

The premium fee is continuously charged to the larger side of the Long and Short positions and is proportionally paid to the other two sides. This mechanism incentivizes market balance by applying a negative funding rate to the sides that take more market risk. Unlikely conventional perpetual exchanges, where the funding rate is manually charged and updated periodically, Derivable's premium is autonomous and continuously applied at every block on the smart contract platform. In particular, counter-party liquidity always receives a portion of premium fee whenever it is applied. Formally, employing another time-decay, we define a compound premium mechanism from payoff functions as followed.

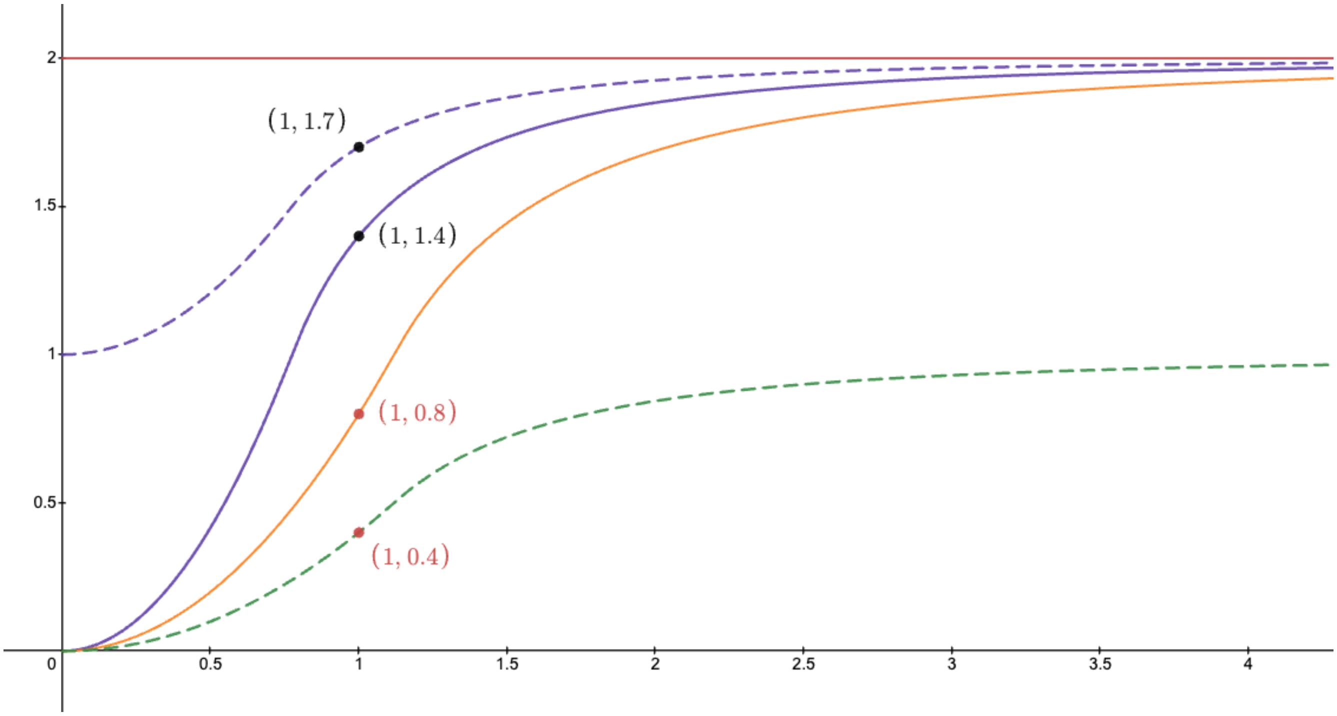


Fig. 6. Half-life decay illustration corresponding with price $x = 1$ and market setting ($R = 2, k = 2, \alpha = 0.8, \beta = 0.6, H = 90$): long value and short value half at $t = H$. Long curves: $\Phi_{t=0}(x)$ (solid green) and half-life $\Phi_{t=H}(x)$ (dashed green). Short curves: $R - \Psi_{t=0}(x)$ (solid violet) and half-life $R - \Psi_{t=H}(x)$ (dashed violet).

Definition 4.5. *Let $\mathcal{M}(k, R, \alpha, \beta, i, \Phi(x), \Psi(x))$ be the current market, t the elapsed time since the previous state, P the premium coefficient. If the market is imbalanced, i.e. $|\Phi(x) - \Psi(x)| > 0$, then the premium fee, $Premium = \max\{\Phi, \Psi\} \times (1 - 2^{\frac{-t}{P}}) \times \frac{|\Phi - \Psi|}{R}$, is applied for new state transitions, explicitly as follows.*

- *If $\Phi(x) > \Psi(x)$, the premium fee will be split between the short side and the counter-party liquidity (LP) pro-rata. The new state transition will be*
  - $\Phi_{new} = \Phi - Premium$
  - $\Psi_{new} = \Psi + Premium \times \dfrac{\Psi}{R - \Phi}$,
  - *implying new counter-party liquidity* $\mathcal{L}_{new} = R - \Phi_{new} - \Psi_{new} = (R - \Phi - \Psi) + Premium \times \dfrac{R - \Phi - \Psi}{R - \Phi}$.
- *If $\Psi(x) > \Phi(x)$, the premium fee will be split between the long side and the counter-party liquidity (LP) pro-rata. The new state transition will be*
  - $\Psi_{new} = \Psi - Premium$
  - $\Phi_{new} = \Phi + Premium \times \dfrac{\Phi}{R - \Psi}$,
  - *implying new counter-party liquidity* $\mathcal{L}_{new} = R - \Phi_{new} - \Psi_{new} = (R - \Phi - \Psi) + Premium \times \dfrac{R - \Phi - \Psi}{R - \Psi}$.

5 Open PnL and Composite Leverage

In Derivable's power perpetual markets, the quantities $\Phi(x), \Psi(x), \Phi(x) + \Psi(x)$ present unsettled or unresolved or open profit & loss (PnL) for long side, short side, and both sides, respectively. These quantities can be considered as equivalency to open interest of order-book and unique position-based derivatives markets.

For each $0 < k \in \mathbb{Z}$, we have a single power leverage for the corresponding market $\mathcal{M}^k$. However, traders want to experience with many power leverage levels as possible, for example, from 2-time to 16-time leverage. It is liquidity inefficiency if constructing 15 markets $\mathcal{M}^k, k = 2, 3, ..., 16$. Thus, to concentrate liquidity, we only construct two markets corresponding with two power leverage levels, for example, $k \in \{2, 16\}$. Then, any leverage between 2 and 16 can be obtained by splitting capital between the two markets.

6 DEX Evolution & Derivable's Differentiation

6.1 On DEX evolution

In order to provide a comprehensive understanding of Derivable's innovation in the decentralized finance (DeFi) space, it is imperative to review the historical evolution of decentralized exchanges (DEXes). DEXes, along with stablecoins, lending services, and trading aggregators, are the critical building blocks of DeFi. However, among these, DEXes have emerged as the most important foundation, serving as the central hub of DeFi. Therefore, this section will provide a succinct overview of the generational advancements of DEXes since 2018, in order to better contextualize the impact of Derivable's innovation. It is important to understand how DEXes have progressed over time in order to fully appreciate the significance of Derivable's contributions.

The 1st generation of DEXes emerged between 2018 and 2019 and relied on the reserve model and limit order book. Under reserve model, a DEX would maintain a reserve of tokens on a contract, enabling traders to exchange one type of token for another based on a price provided by the DEX developer. This model bears similarities to physical foreign-currency exchange shops. Examples of DEXes that utilized this model during this period include Balancer and Kyber in their first versions. However, the reserve model was constrained by liquidity shortages since the reserves were often limited and could even run out. Additionally, these types of DEXes relied on price feed aggregated from centralized exchanges (CEXes). On the other approach, IDEX and Binance DEX attempted to build onchain limit order book but they failed for survival becaused of poor UX caused by the low throughput and high gas fee nature of blockchain.

The 2nd generation of DEXes is based on the constant product or inverse function and liquidity pool model. This model was pioneered by Uniswap, which launched in November 2018 and is founded on a super-simple constant product formula $x * y = k$, where x and y represent cryptocurrencies in a trading pair, and k is a constant parameter. For mathematical formalization, refer to the relevant literature [1]. Uniswap was a game-changing protocol as it introduced the Automated Market Making (AMM) mechanism, which subsequently set the trend in DeFi from 2020 to 2021. Bancor, Balancer and Kyber followed the Uniswap paradigm, completely replaced the reserve model. AMM model allows anyone to become a market maker or liquidity provider, and it is easy, permissionless to create any trading pair on Uniswap and swap any amount instantly, providing an infinitely and permanently available liquidity pool for the swap pair. These features are in contrast to the intrinsic limitations of reserve-type DEXes and centralized exchanges (CEXes). Additionally, the on-chain execution nature of Uniswap and AMM-type DEXes ensures transparency and resistance to censorship. From a mathematical standpoint, this DEX model theoretically offers unlimited liquidity, which is impossible for reserve-type DEXes and CEXes based on limit order-book. However, the Uniswap model (V1 and V2) has limitations, including impermanent loss and slippage, which had never been encountered before.

The 2nd generation of DEXes is still evolving to reduce the impermanent loss (IL) and slippage by concentrating liquidity and swap requests in specific ranges instead of spreading prices on the entire inverse curve. Uniswap V3 led this advancements, followed by other DEXes (Balancer and Kyber, etc). However, liquidity concentration makes the trading experience more complicated and somewhat limits liquidity capacity. An alternative improvement for the constant product formula introduced by Uniswap was the Proactive Market Maker (PMM) algorithm invented by DodoEX. The PMM formula is, in fact, an integral curve of liquidity and price function, offering a more concentrated curve of unlimited liquidity around the current market price. This significantly reduces slippage while multiple the capital efficiency of the liquidity pool. Unfortunately, DodoEX only works if the (average) market price is fed into its markets (via oracle), which is not necessary for Uniswap model and the likes.

Automated Market Maker (AMM) models have rather matured for spot trading, including the Uniswap paradigm, its variations and improvements, and the Proactive Market Maker (PMM) algorithm. While some developers have been trying to reduce the impermanent loss (IL) and slippage through concentration techniques or improving the mathematical curves for liquidity pools, the focus has been shifting to decentralized derivatives exchanges for synthetic assets, perpetual futures and options. This marks the beginning of the **third evolutionary generation** of DEXes. With this advancement, decentralized exchanges are poised to offer a wider range of financial products, making them self-complete and becoming more competitive with centralized exchanges.

In August 2021, a new direction in DeFi was derivatives, pioneered by the dYdX protocol with the introduction of perpetual contract exchange built on StarkWare's StarkEx engine, a permissioned layer-2 solution on Ethereum that utilizes zk-rollup technology. dYdX employs a unique position and limit order book model that is similar to centralized exchanges.

Similarly, some DEXes have been exploring the potential of layer-2 solutions for scalability and gas-saving in developing decentralized perpetual exchanges. GMX, Level Finance have introduced initiatives for decentralized perpetual exchanges that combine liquidity pools, unique positions, and price feeding by the oracle. Instead of a matching engine, GMX and Level Finance uses an oracle to feed the price of underlying assets for the position open and close, liquidation, and risk management. However, liquidity is still limited despite the provision of a liquidity pool. Furthermore, by utilizing an oracle for price-feed, the DEXes indirectly refers to prices on CEXes, hence still relying on order-book pricing mechanisms.

Deri protocol offers perpetual futures, everlasting options, and power perpetual based on NFT-tokenized unique positions and AMM paradigm (adaptive DodoEX's PMM).

Opyn Squeeth, intuitively, introduces fully on-chain, power-perpetual synthetic tokens using on-chain asset over-collateralization, without (unique) positions and order-book model. Mycelium also utilizes power-leverage to offer perpetuals trading and AMM paradigm onchain.

Premia, Dopex, Lyra, Ribbon Finance introduced crypto options to DeFi adopting legacy option pricing models (Black-Scholes, binomial pricing or Monte-Carlo simulation).

6.2 Challenges for derivatives DEXes

Derivatives play a crucial role in the functioning of financial markets, including the DeFi space. Despite their popularity, centralized exchanges have been criticized for their lack of transparency and susceptibility to order book and price manipulation, in particular, bankruptcy (see the collapse of FTX). However, due to their high liquidity, user-friendly interfaces and user-habits, traders often opt for centralized exchanges to trade derivatives such as future options. Several decentralized projects, such as dYdX, GMX, Deri, Level Finance, Premia, Mycelium and Opyn, have emerged in recent years to provide partially decentralized derivative trading solutions. As a result, a growing number of traders and DeFi users have started trading perpetual future contracts and other types of derivatives on decentralized exchanges. We will investigate the key characteristics and limitations of these protocols.

- *dYdX (order book)* is built on StarkEX L2 with restricted future markets, centralized execution, and price feed.
- *GMX, Level Finance (unique position, price feed, and liquidity pool)* is built on Arbitrum One L2, offering limited permits for perpetual future markets. Although settlement is onchain, price feed and risk management of the two protocols are centralized. Compared to dYdX, the liquidity of GMX and Level Finance is enhanced by an open liquidity pool.
- *Premia* takes advantage of liquidity pool to offer AMM-based option trading onchain. However, complicated computation for option pricing based on Black-Scholes model, risk management are still centralized.
- *Deri (AMM paradigm and NFT-position)* offers an open protocol for future and option trading. Deri offers composability, but its NFT-positions imply ill liquidity and low composability.

Despite the introduction of numerous decentralized exchanges (DEXes) for derivatives, offering partial decentralization of certain components, most continue to rely on legacy models for price feed, funding rates, and centralized risk control. To date, no protocol has achieved the level of fully on-chain decentralization exhibited by Uniswap in its execution, price feed, and openness to any trading pair and market maker. Limited liquidity, the determination of suitable funding rates, and systematic risk control mechanisms are challenges for both centralized exchanges and decentralized protocols. To address these challenges, a funding rate should effectively reflect three independent variables: price volatility, long-short deviation, and time, ultimately helping to achieve market equilibrium with practical liquidity. Effective liquidity and risk management strategies are also crucial for the normal operation of derivative markets.

6.3 Differentiating Derivable's solutions

Derivable introduces a novel decentralized derivatives exchange protocol based on a family of asymptotic power curves and Uniswap's automated market maker (AMM) paradigm. Derivable utilizes Uniswap's time-weighted average price TWAP [2] for its price feed, then develops an innovative AMM protocol for power perpetual markets with unique features and properties (see Table 5 for comparison with existing solutions) based on these fundamentals. Major characteristics of Derivable's paradigm are:

- Completely on-chain and open for everyone to participate as liquidity providers (LPs) or traders;
- No order book, no unique position, no liquidation;
- Everlasting with infinite liquidity, even without liquidity providers;
- Liquidity elasticity (impermanent loss and gain) for LPs and leverage elasticity (i.e. leverage vs deleverage with convergence, automatically and continuously adjusted upon market state transitions and price volatility);
- Continuous time-decay and flexible premium mechanism to compensate and protect LP against extreme market volatility with high imbalanced markets.

Protocols	**Trading type**	**Paradigms**	**Execution**	**Price feed**	**Risk control**	**MM**
dYdX	perpetuals	order book	offchain L2	NA	centralized	closed
GMX	perpetuals	price feed, LP	offchain L2	centralized	centralized	closed
Deri	perpetuals	AMM, NFTs	on/off-chain	centralized	decentralized	open
Premia	options	AMM, LP, FT	on/off-chain	centralized	centralized	open
Opyn Squeeth	power perpetuals	LP, position	onchain L1	centralized	decentralized	closed
Mycelium	power perpetuals	AMM, LP, FT	off-chain L2	decentralized	decentralized	open
Derivable	power perpetuals	AMM, LP, FT	onchain L1	decentralized	decentralized	open

Table 5
A comparison among existing derivatives DEXes versus Derivable

6.4 Hedging IL on Uniswap by power perpetuals

Power perpetuals and their applications are attracting interests from both venture capitals (Paradigm) and DeFi projects (Deri Protocol, Predy, Opyn Squeeth). Beside creating and trading power perpetuals in general, a special and meaningful use case of power perpetuals is to hedge IL on Uniswap as presented in the following (readers are referred to Lioba Heimbach et al. [11] for detailed analysis of IL on Uniswap).

Denoting S the price of ETH, x, y the quantities of ETH and USDT, respectively, then the value of liquidity position, the Delta and Gamma of the LP portfolio on Uniswap v2 are:

- $V_{v2} = xS + y = 2\sqrt{SK}$, ($K > 0$ is a given constant),
- Delta $\Delta = \frac{\partial V_{v2}}{\partial S} = \sqrt{\frac{K}{S}}$,
- Gamma $\Gamma = \frac{\partial \Delta}{\partial S} = -\frac{\sqrt{K}}{2S^{3/2}} = -\frac{\sqrt{K}}{2} S^{-3/2}$.

Given a concentration bin (i.e. lower-upper price tick range) $[S_l, S_u]$, the value of liquidity position, the Delta and Gamma of the LP portfolio on Uniswap v3 are:

- $V_{v3} = xS + y = L\left(2\sqrt{S} - \frac{S}{\sqrt{S_u}} - \sqrt{S_l}\right)$, ($L > 0$ is a given constant),
- Delta $\Delta = \frac{\partial V_{v3}}{\partial S} = L(\frac{1}{\sqrt{S}} - \frac{1}{\sqrt{S_u}})$,
- Gamma $\Gamma = \frac{\partial \Delta}{\partial S} = -\frac{L}{2S^{3/2}} = -\frac{L}{2} S^{-3/2}$.

We see that the greeks of LP portfolio on Uniswap v2 & v3 contain square-root terms, especially the Gamma part complicates the impermanent loss. Thus, hedging IL with a linear derivatives (e.g. futures), liquidity providers must rebalance the hedging position frequently to adapt to the rapidly changing Delta. Such a process is called dynamic Delta hedging (DDH), causing a substantial cost. However, it is instantly easy to hedge with power perpetual contracts. Readers are referred to [12], publish0x, and Deri Protocol for details.

Acknowledgement

The author (Thuat Do) expresses a great gratitude to Prof. Yang Wang, vice president of the Hong Kong University of Science and Technology, regarding his great encouragement and support the author's PhD study at HKUST and Blockchain R&D. The authors thank Dr. Bao-Duy Tran for his paper proofreading.

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